APPLICATION OF THE METHOD OF AGGREGATES TO THE SOLUTION OF MIXED PROBLEMS IN ELASTICITY THEORY

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UDC 539.3

The method of aggregates proposed in [1] is as a rule used to solve dynamic problems in elasticity theory. The essence of the method consists of the following. Displacements are expanded in a series in the variable x_i (one series, or two, when boundary conditions are not given on the surfaces x_i = const), and the initial system of dynamical equations is reduced to a series of one-dimensional wave equations, which are solved numerically. In this case, the boundary conditions and the "adhesive" conditions between layers (if these exist) must be uniform.

In this paper, a method is proposed which makes it possible to solve mixed problems.

Lamé's equations in a coordinate system which admits separation of variables are

$$\mu\Delta\mathbf{V} + (\lambda + \mu)\nabla\nabla\cdot\mathbf{V} = \rho\ddot{\mathbf{V}},\ \mathbf{V} = (u,\ w) \tag{1}$$

(for definiteness, the two-dimensional case is chosen). Here u, w are displacements in x_1, x_2; λ, μ are Lamé's parameters; and ρ is the density. Let the equilibrium conditions be given on the coordinate surfaces x_2 = const:

$$\sigma_{12} = \bar{\sigma}_{12},\ \sigma_{22} = \bar{\sigma}_{22} \tag{2}$$

($\{\sigma_{ij}\}$ is the stress tensor; the overbar denotes values along the other side of the boundary), and let

$$\sigma(u,\ \bar{u},\ w,\ \bar{w},\ \sigma_{ij}) = 0,\ \ \tau(u,\ \bar{u},\ w,\ \bar{w},\ \sigma_{ij}) = 0. \tag{3}$$

The choice of functions for σ, τ which make sense physically is not large:

1. Strict contact

$$\sigma = w - \bar{w},\ \tau = u - \bar{u}.$$

2. Contact with discontinuity in the displacements

$$\sigma = \sigma_{22} - P(w - \bar{w}),\ \tau = \sigma_{12} - T(u - \bar{u},\ \sigma_{22})$$

or dilatation

$$\sigma = \sigma_{22} - D(u - \bar{u}),\ \tau = \sigma_{12} - T(u - \bar{u},\ \sigma_{22}),$$

where the functions P, T, and D are in general nonlinear [2].

3. Free surfaces

$$\sigma = \sigma_{22},\ \tau = \sigma_{12}.$$

Let us choose the simplest case, which is a combination of 1 and 3: on part of the surface X ($x_2 = x_2^0$ = const), the boundary is strictly locked, while on the remainder, $\bar{X}$, it is free. We introduce discretization of x_2 with step size h and write these conditions and (2), taking into account Hooke's law, in the form

$$\mu\left(\frac{u - u_*}{h} + \frac{\partial w}{\partial x_1}\right) = \bar{\mu}\left(\frac{\bar{u}_* - \bar{u}}{h} + \frac{\partial \bar{w}}{\partial x_1}\right),\ \ u = \bar{u},$$

Novosibirsk. Translated from Prikladnaya Mekhanika i Tekhnicheskaya Fizika, No. 5, pp. 106-110, September-October, 1992. Original article submitted July 12, 1991.

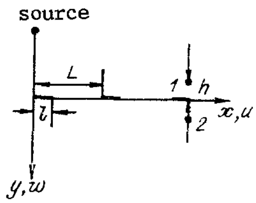

Fig. 1

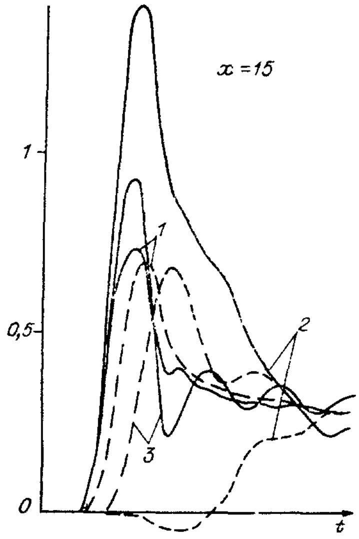

Fig. 2

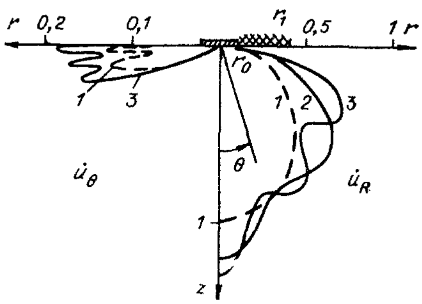

Fig. 3

$$\alpha\frac{w-w_*}{h}+\lambda\frac{\partial u}{\partial x_1}=\bar{\alpha}\frac{\bar{w}_*-\bar{w}}{h}+\bar{\lambda}\frac{\partial\bar{u}}{\partial x_1},\quad w=\bar{w}\ \text{for}\ x_1\in X;$$

$$\mu\left(\frac{u-u_*}{h}+\frac{\partial w}{\partial x_1}\right)=\bar{\mu}\left(\frac{\bar{u}_*-\bar{u}}{h}+\frac{\partial\bar{w}}{\partial x_1}\right)=0,$$

$$\alpha\frac{w-w_*}{h}+\lambda\frac{\partial u}{\partial x_1}=\bar{\alpha}\frac{\bar{w}_*-\bar{w}}{h}+\bar{\lambda}\frac{\partial\bar{u}}{\partial x_1}=0\ \text{for}\ x_1\in\bar{X};$$

$$u_*=u(x_1,x_2-h),\quad \bar{u}_*=\bar{u}(x_1,x_2+h),\quad \alpha=\lambda+2\mu,$$

from which we have, for $x_1 \in X$

$$u=\bar{u}=bu_*+\bar{b}\bar{u}_*+h(\bar{b}-b)\frac{\partial w}{\partial x_1},$$
$$w=\bar{w}=aw_*+\bar{a}\bar{w}_*+h\frac{\bar{\lambda}-\lambda}{\bar{\alpha}+\alpha}\frac{\partial u}{\partial x_1}, \tag{4}$$
$$b=\mu/(\bar{\mu}-\mu),\ \bar{b}=1-b,\ a=\alpha/(\bar{\alpha}+\alpha),\ \bar{a}=1-a,$$

while for $x_1 \in \bar{X}$

$$u=u_*-h\frac{\partial w}{\partial x_1},\quad w=w_*-\delta h\frac{\partial u}{\partial x_1}, \tag{4'}$$
$$\bar{u}=\bar{u}_*+h\frac{\partial\bar{w}}{\partial x_1},\quad \bar{w}=\bar{w}_*+\bar{\delta}h\frac{\partial\bar{u}}{\partial x_1},\quad \delta=\lambda/\alpha,\quad \bar{\delta}=\bar{\lambda}/\bar{\alpha}.$$

Let

$$u=\sum_{n=0}^{N}U^n(x_2)S_n(x_1),\quad w=\sum_{n=0}^{N}W^n(x_2)C_n(x_1) \tag{5}$$

($\{S_n, C_n\}$ is a complete system of functions).

For a Cartesian coordinate system ($x_1 = x$, $x_2 = y$)

$$\{S_n(x),\ C_n(x)\}=\{\sin k_nx,\ \cos k_nx\},\ k_n=\pi n/A$$

($x = A$ is a ficticious boundary [1]).

For a cylindrical coordinate system ($x_1 = \theta$, $x_2 = r$, $A = \pi$)

$$\{S_n(\theta),\ C_n(\theta)\}=\{\sin n\theta,\ \cos n\theta\}$$

($x_1 = r$, $x_2 = z$), $S_n(r) = J_1(k_nr)$, $C_n(r) = J_0(k_nr)$ (in this case, k_0 are roots of the equation $J_1(k_nA) = 0$, and J_0 and J_1 are Bessel functions).

For a spherical coordinate system ($x_1 = \varphi$, $x_2 = R$, $A = \pi$)

$$S_n(\varphi) = P_n^1(\cos\varphi), \quad C_n(\varphi) = P_n^0(\cos\varphi)$$

(P_n^0, P_n^1 are Legendre polynomials).

We now apply the operation $\int_0^A \ldots S_n(x_1)\,dx_1$ to the first relations in (4) and (4'), and the operation $\int_0^A \ldots C_n(x_1)\,dx_1$ to the second relations in (4) and (4'), using the necessary weight functions in the operations. After transformation, we obtain a system of linear equations for U^n, W^n, $\bar{U}^n$, $\bar{W}^n$:

$$\begin{gathered}
U^m = U_*^m - p_m W^m + \bar{b} \sum_{n=0}^{N} (Q_*^n + 2p_n W^n) A_{mn}, \\
W^m = W_*^m - \delta p_m U^m + \bar{a} \sum_{n=0}^{N} (T_*^n + c p_n U^n) B_{mn}, \\
\bar{U}^m = \bar{U}_*^m + p_m \bar{W}^m - b \sum_{n=0}^{N} (Q_*^n + 2p_n \bar{W}^n) A_{mn}, \\
\bar{W}^m = \bar{W}_*^m + \bar{\delta} p_m \bar{U}^m - a \sum_{n=0}^{N} (T_*^n + c p_n \bar{U}^n) B_{mn}, \qquad (6) \\
Q_*^n = \bar{U}_*^n - U_*^n, \quad T_*^n = \bar{W}_*^n - W_*^n, \quad c = \delta + \bar{\delta}, \quad m = 0, \ldots, N, \quad p_n = hk_n, \\
A_{mn} = \int_0^A g(x_1) S_m(x_1) S_n(x_1)\,dx_1, \quad B_{mn} = \int_0^A g(x_1) C_m(x_1) C_n(x_1)\,dx_1, \\
g(x_1) = \begin{cases} 1 & x_1 \in X, \\ 0 & x_1 \in \bar{X}. \end{cases}
\end{gathered}$$

The limiting cases are $A_{mn} = B_{mn} = 0$ (a free surface); and $A_{mn} = B_{mn} = \delta_{mn}$ (strict contact). The solution to (6) does not present any difficulties.

If condition (3) is nonlinear, then to determine the displacement at $x_2 = x_2^0$, it is necessary to iterate at each time step.

We examine two cases using the proposed approach.

1. Qualitative Monitoring of the State of Contact between Blocks. It has been shown in a number of works (for example, [3]) that the contact of two blocks of rock does not take place along the entire surface, but only over an insignificant part of the surface. Then, with increasing normal compressive stress σ_n, the contact area increases. (The contact area is defined as the total area of the "spots" of contact.) Recall that the tangential contact strength is $\tau_* = |\sigma_n|\tan\psi_* + \tau_0$ (ψ_* is the "angle of internal friction," τ_0 is the adhesion), and when τ_* is exceeded, dynamic phenomena (the release of stored energy) become possible. Using this information, it is possible to judge the tendencies in interblock contact behavior based on the character of the change in σ_n, and, if observations are made at some cracks, to judge the behavior of sections of the block as a whole.

Thus, let a pulsed point source of the expansion–contraction type be located in the neighborhood of a crack, the margins of which are in contact along some segment (Fig. 1). Unidirectional displacement gauges are attached along both sides of the crack at a vertical distance h from the crack. The source generates a wave whose wavelength Λ is much greater than the dimension ℓ of the contact segment. We considered just such a case above.

Computational results are shown in Fig. 2. The displacement at the upper observation point is given by solid curves; that at the lower point by dashed curves. The maximum amplitude of the vertical displacement in a square wave at the point (0, 0) was chosen as unity. Curves 1-3 correspond to $\zeta = L/\ell = \infty$, 10, 2 ($\zeta = \infty$ is the case of strict contact). The signal radiating from the source is one period of a sinusoid. Here it is plainly evident that with decreasing contact area, the amplitude of the first arrival grows at point 1, but drops at point 2 (Fig. 1). This difference is significantly reduced with the passage of time. The amplitude at the upper point exceeds unity, due to the presence of free segments (it is known that the displacement is doubled at a free surface).

Note that the observation point must be situated at a distance less than Λ from the line of contact. In the opposite case, all useful information is lost as a consequence of the superposition of waves. The proposed scheme is easily implemented in practice and makes it possible to qualitatively monitor the ongoing state of a crack. The ratio of the amplitudes of the first arrivals at point 1 and 2 serves as an informative parameter.

To obtain quantitative estimates, it is necessary to experimentally investigate the connection between normal forces and the contact area (of the Hertz problem type).

2. Method for Signal Amplification from an Oscillator Source. It is known that during the operation of a surface oscillatory source, longitudinal waves carry only 7 to 15% of the energy [4]. We examine one of the methods for amplifying the signal in the longitudinal direction: Part of the surface around the source is artificially suppressed in the vertical direction. Without going into technical details, we write the boundary conditions at the surface $x_2 = 0$ (Fig. 3):

$$\sigma_{yy} = \begin{cases} -F(t) & x_1 \leqslant r_0, \\ 0 & x_1 > r_1, \end{cases} \quad w = 0 \quad r_0 < x_1 \leqslant r_1, \quad \tau = 0.$$

Here $F(t) = \sin \omega t \varepsilon(t)$; ε is the Heaviside function; $\omega = 2\pi f$; f is the frequency; r_0 and r_1 are the dimensions of the source and of the suppressed region. The problem is solved in a cylindrical coordinate system.

In accordance with the approach outlined above, we can immediately write out the relations

$$\sum_{i=0}^{N} T_{mi} W^i = W_0^m, \quad U^m = U_*^m - p_m W^m, \quad m = 0, \ldots, N,$$

$$T_{mi} = \left(1 + \delta p_m^2\right) \delta_{mi} - \delta p_i^2 C_{mi},$$

$$W_0^m = W_*^m + \delta p_m U_*^m + Q_m - W_*^0 C_{m0} - \sum_{i=1}^{N} \left(W_*^i + \delta p_i U_*^i\right) C_{mi},$$

$$C_{mi} = \int_0^A g(x_1) x_1 J_0(k_m x_1) J_0(k_i x_1)\, dx_1,$$

$$Q_m = 2F(t) J_1(k_m r_0) / \left(k_m J_0^2(k_m A)\right),$$

$$g(x_1) = \begin{cases} 1 & \text{for } r_0 < x_1 \leqslant r_1, \\ 0 & \text{for } 0 \leqslant x_1 \leqslant r_0,\ r_1 < x_1. \end{cases}$$

The calculations were done for the following values of the dimensionless parameters: $\eta = 0.033$, $\gamma = 0.3$ ($\eta = fr_0/V_p$, $\gamma = V_s/V_p$, V_p and V_s are the longitudinal and transverse wave velocities).

Figure 3 shows cylindrical-coordinate plots of the radial and tangential displacement velocities (the maximum value of $\dot{u}_R$ at $\xi = 0$, where $\xi = r_1/r_0 - 1$ was chosen as unity) $\dot{u}_R = u \times \sin\theta + \dot{w}\cos\theta$, $\dot{u}_\theta = \dot{u}_\theta = \dot{u}\cos\theta - \dot{w}\sin\theta$ for $\xi = 0, 4, 9$ (curves 1-3). Analysis of these plots allows us to conclude that suppression of the surface has a positive effect. Depending on the values of ξ, η, and γ, the amplitude of the radial velocities can be increased by 30% or more. With decreasing γ, focussing in the vertical direction is improved, and the $\dot{u}_\theta$ component is strongly suppressed.

In conclusion, note that the proposed method is not useful for studying "fine" effects, as in the analysis of stress concentrations at singular points (for example, $x_2 = r_0$ in problem 2; here the factorization method is used [5]). This is because it is necessary to select a finite upper limit for the summations in (5). The method is effective for making global estimates of the parameters of the process.

LITERATURE CITED

1. B. G. Mikhailenko, "Unsteady seismic fields in nonuniform media (a numerical investigation of wave dynamics)," Author's abstract of a doctoral dissertation in the physico-mathematical sciences, Novosibirsk (1982).
2. A. F. Revuzhenko, "A variational statement of boundary-value problems of failure," Prikl. Mekh. Tekh. Fiz., No. 4 (1980).
3. S. R. Brown and C. H. Scholz, "Closure of random elastic surfaces in contact," J. Geophys. Res., 90, No. B7 (1985).
4. V. T. Grinchenko and V. V. Meleshko, Harmonic Oscillations and Waves in Elastic Bodies [in Russian], Naukova Dumka, Kiev (1981).
5. V. Z. Parton and P. I. Perlin, Methods in Mathematical Elasticity Theory [in Russian], Nauka, Moscow (1981).

EFFECT OF A PLANE ACOUSTIC PRESSURE WAVE ON A REINFORCED CYLINDRICAL SHELL

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UDC 539.3

An investigation of the strength of reinforced shells against pulsed loads is required to determine the limits of applicability of various designs in mechanical engineering and construction. This has resulted in a large number of publications on the development of theory and calculational method for ribbed shells (see review [1]). The effect of reinforcement ribs on the stress–strain state and the kinematic fields of cylindrical shells immersed in a fluid has been examined for transient excitation [2-4]. Special attention has been paid [2] to membrane stresses in the central cross section under the action of a plane wave. Radial displacements have been investigated [3] for axisymmetric loading in the center of the shell. Interaction of the fluid with the shell has been studied [4] according to the hypothesis of plane reflection. The behavior of flexure stresses on reinforced shells has hardly been studied.

Here we estimate the flexure and membrane stresses and the displacements of periodically reinforced shells during the transverse action of a plane translational pressure wave. A numerical solution of the problem is obtained by using a Fourier expansion in the angular coordinate and by using finite differences in the other coordinates. Numerical and analytical results are compared. The dynamic-response factor and the initial time at which these results coincide are determined.

1. The transient effect of a plane translational pressure wave is investigated for an infinitely long, thin, elastic cylindrical shell, which is periodically reinforced by ribs and immersed in an ideal elastic fluid. The shell is either empty or filled with the same fluid as surrounds it. The front of the incoming wave is parallel to the axis of the shell. The movement of the shell is described by linear equations of the Kirchhoff–Love theory; the excitations in the fluid are described by the wave equation for the velocity potential. The equations of motion for the m-th mode of oscillations along the angle θ have the form

$$\frac{1}{c^2}\frac{\partial^2 u_m}{\partial t^2} = \frac{\partial^2 u_m}{\partial x^2} - \frac{1-\nu}{2}\frac{m^2}{R^2}u_m + \frac{1+\nu}{2}\frac{m}{R}\frac{\partial v_m}{\partial x} + \frac{\nu}{R}\frac{\partial w_m}{\partial x},$$

$$\frac{1}{c^2}\frac{\partial^2 v_m}{\partial t^2} = \frac{1-\nu}{2}\frac{\partial^2 v_m}{\partial x^2} - \frac{1+\nu}{2}\frac{m}{R}\frac{\partial u_m}{\partial x} - \frac{m^2}{R^2}v_m - \frac{m}{R^2}w_m +$$

$$+ \frac{\delta^2}{12R^2}\left\{2(1-\nu)\frac{\partial^2 v_m}{\partial x^2} - \frac{m^2}{R^2}v_m - \frac{m^3}{R^2}w_m + (2-\nu)\,m\frac{\partial^2 w_m}{\partial x^2}\right\},$$

$$\frac{1}{c^2}\frac{\partial^2 w_m}{\partial t^2} = -\frac{\nu}{R}\frac{\partial u_m}{\partial x} - \frac{m}{R^2}v_m - \frac{w_m}{R^2} - \frac{\delta^2}{12}\left[\frac{\partial^4 w_m}{\partial x^4} - \frac{2m^2}{R^2}\frac{\partial^2 w_m}{\partial x^2} + \right.$$

Novosibirsk. Translated from Prikladnaya Mekhanika i Tekhnicheskaya Fizika, No. 5, pp. 110-116, September-October, 1992. Original article submitted March 29, 1991; revision submitted June 20, 1991.